

On the wildness of cambrian lattices

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Abstract

In this note, we investigate the representation type of the cambrian lattices and some other related lattices. The result is expressed as a very simple trichotomy. When the rank of the underlying Coxeter group is at most 2, the lattices are of finite representation type. When the Coxeter group is a reducible group of type \mathbb{A}_1^3 , the lattices are of tame representation type. In all the other cases they are of wild representation type.

1 Introduction

By a famous result of Y. Drozd [Dro80], associative algebras can be classified according to their representation type, which can be finite, tame or wild. Algebras of finite representation type have a finite number of isomorphism classes of indecomposable modules. They are quite rare in some sense, and a random algebra, meaning roughly any sufficiently complicated algebra, should be expected *a priori* to be wild, unless there is a specific reason for the contrary. Algebras of tame type are living on the frontier between finite type and wild type, and should not be expected to appear frequently either.

Given any finite poset P and a base field \mathbf{k} , one can define the incidence algebra of P over \mathbf{k} , which is a finite dimensional associative algebra of finite global dimension. It is a natural question to ask what is the representation type of incidence algebras. A general description of the representation-finite case has been made in [Lou75b, Lou75a]. The tame case has been considered in [Les03, Les02].

In this short note, an answer is given for some families of posets, namely the cambrian lattices and some other related lattices. Cambrian lattices have been introduced by N. Reading [Rea06, Rea07] to provide a description of the most combinatorial aspects of the theory of cluster algebras of S. Fomin and A. Zelevinsky. More than that, these posets allow to extend this combinatorial description from finite root systems to finite Coxeter groups, therefore encompassing the non-crystallographic finite Coxeter groups of type \mathbb{I} and \mathbb{H} .

Our main aim in this note is to tell which of the cambrian lattices have finite representation type, and which have tame or wild representation type. It turns out that for this family of posets the trichotomy is particularly simple: when the rank of W is 1 or 2, then the cambrian lattices are of finite representation type. Otherwise they are of wild representation unless they are isomorphic to the cube which is tame. This last case arises exactly when $W = \mathbb{A}_1 \times \mathbb{A}_1 \times \mathbb{A}_1$.

In the remaining part of the note, the same result is obtained for two related families, namely lattices of order ideals of root posets and Stokes lattices.

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2 Representation theory of posets

The incidence algebra of a poset P over a field \mathbf{k} has a basis $I_{a,b}$ indexed by pairs of comparable elements $a \leq b$ in P and its associative product is defined on this basis by

$$I_{a,b}I_{c,d} = \begin{cases} 0 & \text{if } b \neq c, \\ I_{a,d} & \text{if } b = c. \end{cases} \quad (1)$$

Every poset can be seen as a small category, where there is a (unique) map from a to b if and only if $a \leq b$ in P . The category of finite-dimensional modules over the incidence algebra of P over \mathbf{k} is equivalent to the category of functors from the poset P to the category of finite-dimensional vector fields over \mathbf{k} , denoted by $\mathcal{F}_{P,\mathbf{k}}$.

This can also be restated, using the Hasse diagram of the poset P , as the category of representations of a quiver with relations. The arrows $a \rightarrow b$ of the quiver are the cover relations $a \triangleleft b$ in the poset P , and the relations are commuting relations, saying that any two paths sharing their ends are equal.

By a slight abuse of notation, we will say that two posets P and Q are derived equivalent when the derived categories of modules over their incidence algebras are triangle-equivalent.

Similarly, we will say that P has finite, tame or wild representation type if the incidence algebra of P has this property.

In the rest of this note we assume that \mathbf{k} is an algebraically closed field. We will systematically use the following well-known Lemma. Recall that a subposet of a poset Y is the induced partial order on a subset of Y .

Lemma 2.1. *Let X be a subposet of a finite poset Y .*

1. *If X is of wild representation type, then Y is of wild representation type.*
2. *If X is of tame representation type, then Y is either of tame or of wild representation type.*

Proof. We view X and Y as finite categories. Then, the embedding i of X in Y is a fully-faithful functor. The precomposition by this functor induces a functor i^{-1} from $\mathcal{F}_{Y,\mathbf{k}}$ to $\mathcal{F}_{X,\mathbf{k}}$. By usual arguments, it has a left adjoint $i_!$ given by the left Kan extension along i . If $F \in \mathcal{F}_{X,\mathbf{k}}$, then by Proposition 3.7.3 of [Bor94], we have $(i^{-1} \circ i_!)(F) \cong F$. It follows that the functor $i_!$ sends an indecomposable object of $\mathcal{F}_{X,\mathbf{k}}$ to an indecomposable object of $\mathcal{F}_{Y,\mathbf{k}}$. Moreover, for F_1 and F_2 in $\mathcal{F}_{X,\mathbf{k}}$, we have $i_!(F_1) \cong i_!(F_2)$ if and only if $F_1 \cong F_2$. \square

We will also need the following well-known result.

Proposition 2.2. *Let X and Y be two finite posets and $f : X \rightarrow Y$ be a surjective morphism of posets such that $f^{-1}(y)$ is connected for every $y \in Y$. Then, if Y is wild representation type, so is X .*

Proof. This is Proposition 1.3 of [Lou75a]. This follows from the fact that the functor $f^{-1} : \mathcal{F}_{Y,\mathbf{k}} \rightarrow \mathcal{F}_{X,\mathbf{k}}$ is a fully-faithful embedding. \square

Derived equivalences provide powerful tools for the study of finite dimensional algebras and finite posets. When two algebras share the same derived categories, they share a number of invariants such as the number of isomorphism classes of simple modules, or the center. However, the representation type is not preserved by derived equivalences. Still, when a finite dimensional algebra is derived equivalent to a hereditary algebra, its representation type is dominated¹ by the representation type of the hereditary algebra.

Proposition 2.3. *Let A be a finite dimensional \mathbf{k} -algebra. Let \mathcal{H} be an abelian hereditary category. If $D^b(A) \cong D^b(\mathcal{H})$, then the representation type of A is dominated by the representation type of \mathcal{H} .*

¹Here dominated means smaller in the order where (finite) \leq (tame) \leq (wild).

Proof. Let $F : D^b(A) \rightarrow D^b(\mathcal{H})$ be an equivalence of triangulated categories. As usual, we view the indecomposable A -modules as complexes concentrated in degree zero. Then, the functor F sends an indecomposable A -module to an indecomposable object of $D^b(\mathcal{H})$. Moreover, there exist two integers m and n , independent of X (but which depend on the choice of the equivalence F) such that the homology of $F(X)$ is concentrated in degrees in $[m, n]$. Since the category \mathcal{H} is hereditary, the indecomposable objects of $D^b(\mathcal{H})$ are just shifts of indecomposable objects of \mathcal{H} . So the images of the indecomposable A -modules under F lie in a finite number of copies of \mathcal{H} . For more details, see for example Section 2 of [HZ10]. \square

Proposition 2.4 (Ladkani). *Let X be a finite poset with a unique maximal element $\hat{1}$. Let $Y = X \setminus \{\hat{1}\} \sqcup \{\hat{0}\}$ where $\hat{0}$ is the unique minimal element of Y . Then, the posets X and Y are derived equivalent.*

Proof. This is a special case of the much more general *flip-flop* of Ladkani (see Corollary 1.3 of [Lad07a]). In this simple case, the derived equivalence can be realized by the following tilting complex. If $x \in X$, we denote by P_x the projective indecomposable module corresponding to the element x . If $x \neq \hat{1}$, we consider the complex $T_x = P_{\hat{1}} \xrightarrow{f} P_x$, where f is the canonical embedding of $P_{\hat{1}}$ into P_x and $P_{\hat{1}}$ is in degree zero of the complex. Then, $T = P_{\hat{1}} \bigoplus_{\hat{1} \neq x \in X} T_x$ is a tilting complex and its endomorphism algebra is the incidence algebra of Y . \square

Proposition 2.5. *Let \mathbf{k} be an algebraically closed field. Let $n \geq 3$. Let Y be a finite poset whose Hasse diagram is a non circular orientation of an affine diagram of type $\mathbb{A}_n^{(1)}$. Let $X = Y \sqcup \{\omega\}$ be a poset whose Hasse diagram contains exactly two more edges, in such a way that ω is the maximal element or the minimal element of a commutative square. Then, X is of wild representation type.*

Proof. Let us remark that it is always possible to produce such a poset X . The orientation of the diagram $\mathbb{A}_n^{(1)}$ is non circular, so there are subposets of $\mathbb{A}_n^{(1)}$ of the form $\bullet \xrightarrow{a} \bullet \leftarrow \bullet \xrightarrow{b}$ and of the form $\bullet \xleftarrow{c} \bullet \rightarrow \bullet \xleftarrow{d}$. If we set $\omega > a$ and $\omega > b$ we have a poset X such that ω is the maximal element of a commutative square. If we set $\omega < c$ and $\omega < d$, then ω is the minimal element of a commutative square. By duality, it is enough to prove the result when ω is the minimal element of a commutative square. We exhibit a two-parameter family of pairwise non-isomorphic indecomposable representations for the poset X inspired by a similar construction for the preprojective algebra of type A_6 (see [Ric15]). This implies that X is of wild representation type.

Let $i : \mathbb{A}_n^{(1)} \rightarrow X$ be the canonical embedding. By precomposition, we have a functor $i^{-1} : \mathcal{F}_{X, \mathbf{k}} \rightarrow \mathcal{F}_{\mathbb{A}_n^{(1)}, \mathbf{k}}$ which is nothing but the obvious restriction to the subposet $\mathbb{A}_n^{(1)}$.

We fix α an arrow of the Hasse diagram of X which is not in the commutative square. Let $\lambda \in \mathbf{k}$. For $x \in \mathbb{A}_n^{(1)}$, we set $M(\lambda)(x) = \mathbf{k}$. We let $M(\lambda)(\alpha) : \mathbf{k} \rightarrow \mathbf{k}$ be the multiplication by λ . And for every arrow α' in the Hasse diagram of $\mathbb{A}_n^{(1)}$, we let $M(\lambda)(\alpha') = \text{Id}_{\mathbf{k}}$. Extending $M(\lambda)$ to any morphism of the finite category $\mathbb{A}_n^{(1)}$, we have a functor $M(\lambda)$ in $\mathcal{F}_{\mathbb{A}_n^{(1)}, \mathbf{k}}$. For $\lambda, \mu \in \mathbf{k}$, it is easy to see that

$$\text{Hom}_{\mathcal{F}_{\mathbb{A}_n^{(1)}, \mathbf{k}}} (M(\lambda), M(\mu)) \cong \begin{cases} \mathbf{k} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

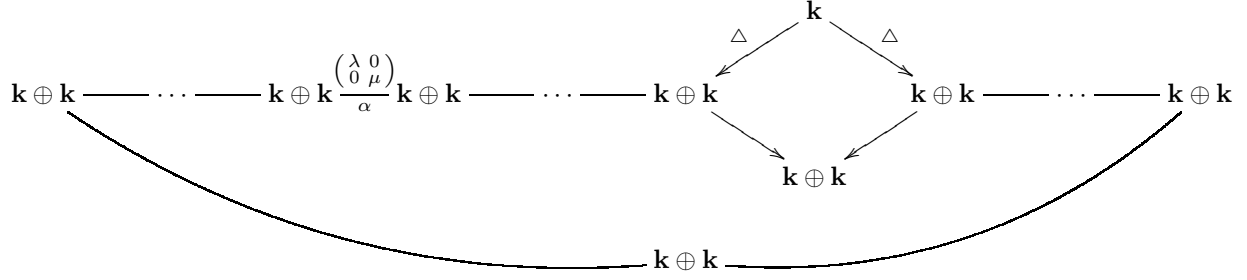
Here, the isomorphism $\mathbf{k} \cong \text{End}(M(\lambda))$ is given by the application that sends $t \in \mathbf{k}$ to the natural transformation $(t_x)_{x \in \mathbb{A}_n^{(1)}} \in \text{End}(M(\lambda))$, where $t_x : M(\lambda)(x) = \mathbf{k} \rightarrow M(\lambda)(x) = \mathbf{k}$ is the multiplication by t . Since the functor $M(\lambda)$ has a local endomorphism algebra, it is indecomposable.

Let $\lambda, \mu \in \mathbf{k}$ such that $\lambda \neq \mu$. Then, we consider the functor $M(\lambda, \mu)$ in $\mathcal{F}_{X, \mathbf{k}}$ defined as follows:

$$i^{-1}(M(\lambda, \mu)) = M(\lambda) \oplus M(\mu), \text{ and } M(\lambda, \mu)(\omega) = \mathbf{k}.$$

For the two arrows g_1 and g_2 starting at ω , we let $M(\lambda, \mu)(g_i)$ be the diagonal embedding \triangle of \mathbf{k} into $\mathbf{k} \oplus \mathbf{k}$.

In other terms, the representation $M(\lambda, \mu)$ has the following shape:



where all the non-labelled edges correspond to the identity matrix.

Let f be an endomorphism of $M(\lambda, \mu)$. Then, $i^{-1}(f)$ is an endomorphism of $M(\lambda) \oplus M(\mu)$. In particular, because one assumes that $\lambda \neq \mu$, it is of the form $f_1 \oplus f_2$ where f_1 is an endomorphism of $M(\lambda)$ and f_2 is an endomorphism of $M(\mu)$. Moreover, we know that f_1 (resp. f_2) is the natural transformation given by the multiplication by an element $t_1 \in \mathbf{k}$ (resp. $t_2 \in \mathbf{k}$). It follows that for every $x \in \mathbb{A}_n^{(1)}$, we have $f_x = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$. If we look at the situation at the vertex ω , we have the following commutative square:

$$\begin{array}{ccc} \mathbf{k} & \xrightarrow{f_\omega} & \mathbf{k} \\ \downarrow \Delta & & \downarrow \Delta \\ \mathbf{k} \oplus \mathbf{k} & \xrightarrow{f_1 \oplus f_2} & \mathbf{k} \oplus \mathbf{k} \end{array}$$

So, we have $t_1 = t_2 = f_\omega(1)$. It is now clear that $\text{End}(M(\lambda, \mu)) \cong \mathbf{k}$. In particular, the functor $M(\lambda, \mu)$ is indecomposable. Let $\lambda' \neq \mu' \in \mathbf{k}$. If there is an isomorphism $f : M(\lambda, \mu) \rightarrow M(\lambda', \mu')$, then by restriction, there is an isomorphism from $M(\lambda) \oplus M(\mu)$ to $M(\lambda') \oplus M(\mu')$. This implies that $\{\lambda, \mu\} = \{\lambda', \mu'\}$. Since the roles of λ and μ are symmetric, there is an isomorphism $M(\lambda, \mu) \cong M(\mu, \lambda)$. In other words, there is an isomorphism between the indecomposable representations $M(\lambda, \mu)$ and $M(\lambda', \mu')$ if and only if $\{\lambda, \mu\} = \{\lambda', \mu'\}$. \square

3 Cambrian lattices

Let W be a finite Coxeter group. Let n denote the rank of W , which is the number of simple generators $\{s_1, \dots, s_n\}$.

When the Coxeter graph of W is connected, so that W is not a product of smaller Coxeter groups, we will say that W is irreducible, and reducible otherwise.

Recall that a Coxeter element c in W is the product of the simple generators in some order.

Once a Coxeter element c is chosen, one can define the cambrian lattice $\text{Cambrian}(W, c)$ using restriction of the weak order on W to the set of c -sortable elements. The reader is referred to the original articles [Rea06, Rea07] for the precise definition.

For a given W , the cambrian lattices for various choices of c share the same number of elements, which is called the Coxeter-Catalan number of type W . Much more is true, namely their Hasse diagrams are orientations of the same unoriented n -regular graph, which is the exchange graph of cluster theory when W is crystallographic.

Something deeper also holds, at least in the case of simply-laced crystallographic W . S. Ladkani has proved in [Lad07b], in the much more general case of quivers, the following statement.

Proposition 3.1. *Let W be a simply-laced crystallographic root-system. Let c and c' be two Coxeter elements for W . Then the incidence algebras of $\text{Cambrian}(W, c)$ and $\text{Cambrian}(W, c')$ are derived equivalent.*

This proposition is expected to hold in all cases, but has not been proved yet, to the best of our knowledge.

3.1 Easy types

There are two situations when one can easily determine the representation type of the poset $\text{Cambrian}(W, c)$.

Lemma 3.2. *If the rank n of W is at most 2, every poset $\text{Cambrian}(W, c)$ is of finite representation type.*

Proof. In rank 1, this is trivial for the quiver with one vertex.

In rank 2, for the Coxeter groups of type $\mathbb{I}_2(h)$, the Hasse diagram of $\text{Cambrian}(W, c)$ is obtained by adding a minimal element and a maximal element to the disjoint union of the chain poset of size $h - 1$ and the poset with one element.

By Proposition 2.4, the poset $\text{Cambrian}(W, c)$ is derived equivalent to a poset which is just a quiver, and has type \mathbb{D}_{h+2} . This quiver is therefore of finite representation type, and the conclusion follows by Proposition 2.3. \square

If the rank n is at least 4, then one can proceed as follows. Recall that the cambrian lattice of a Coxeter group W of rank n has a n -regular Hasse diagram.

Lemma 3.3. *Let P be a lattice whose Hasse diagram is a 4-regular graph. Then P is of wild representation type.*

Proof. Let a be the unique minimal element of P and let b_1, b_2, b_3, b_4 be the elements that cover a .

Assume first that some b_i is covered by an element c that does not cover any other b_j . The induced quiver on the set $\{a, b_1, b_2, b_3, b_4\}$ has affine type $\mathbb{D}_4^{(1)}$. Then, the induced quiver on the set $\{a, b_1, b_2, b_3, b_4, c\}$ has wild representation type.

Otherwise, every element covering one of the b_i covers at least another b_j . Because P is a lattice, for every pair $1 \leq i < j \leq 4$, there is at most one element $c_{i,j}$ covering both b_i and b_j . So in particular there are at most 6 elements covering one of the b_i . Using the hypothesis that the Hasse diagram is 4-regular, one can deduce that, for every $1 \leq i < j \leq 4$, there exists an element $c_{i,j}$ covering exactly both b_i and b_j . Note that there can be no arrows between the $c_{i,j}$.

Now consider the subquiver on the set $\{b_1, b_2, b_3, c_{1,2}, c_{2,3}, c_{1,3}, c_{1,4}\}$. Without $c_{1,4}$, this would be an affine quiver of type $\mathbb{A}_5^{(1)}$. This is therefore a quiver of wild representation type. \square

Lemma 3.4. *Let P be a poset that has a minimal element a covered by $n \geq 5$ elements b_1, \dots, b_n . Then P is of wild representation type.*

Proof. The induced quiver on $\{a, b_1, \dots, b_n\}$ has wild representation type because $n \geq 5$. \square

In the case of rank $n = 3$, the situation is less clear. By picking appropriate subsets of vertices, we will now show that the posets $\text{Cambrian}(W, c)$ are wild unless they are isomorphic to the cube.

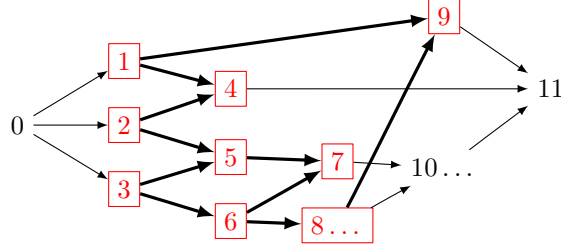
3.2 Reducible types of rank 3

Lemma 3.5 (Lenzing). *The cube poset is of tame representation type.*

Proof. The subposet of the cube consisting of all the vertices except the minimal and the maximal elements of the cube is a quiver of type $\mathbb{A}_5^{(1)}$. By Lemma 2.1, the cube has an infinite representation type. Moreover, by Example 18.6.2 of [Len99] or by Example 2.2 of [Lad07c], the incidence algebra of the cube is derived equivalent to the weighted projective line $\mathbb{X}(3, 3, 3)$. By Section 4 of [Len92], the category $\text{coh}(\mathbb{X}(3, 3, 3))$ is tame. Since it is a hereditary category, the result follows from Proposition 2.3. \square

Proposition 3.6. *Let $h \geq 3$. Then, the poset $\text{Cambrian}(W_{\mathbb{A}_1 \mathbb{I}_2(h)}, c)$ is wild.*

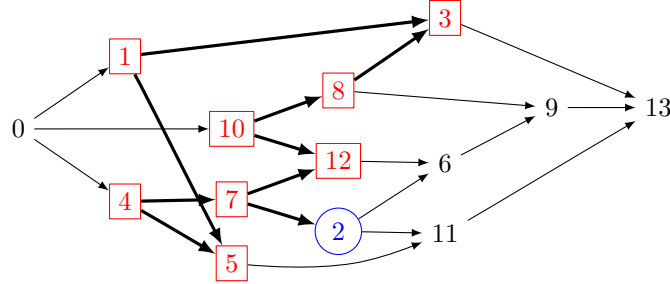
Proof. The poset attached to the reducible type $\mathbb{A}_1\mathbb{I}_2(h)$ is the cartesian product of a segment and a $(h-1, 1)$ -cycle: a poset obtained by adding a minimal element and a maximal element to the disjoint union of the chain poset of size $h-1$ and the poset with one element. We exhibit a subposet which has wild representation type by Proposition 2.5.



□

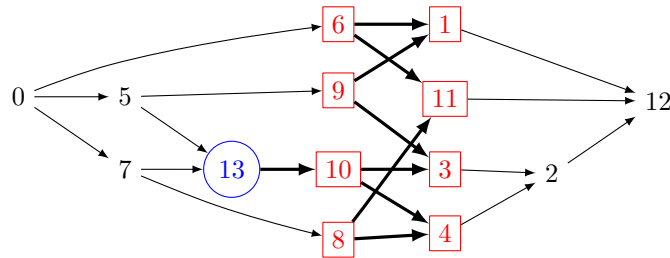
3.3 Type \mathbb{A}_3

Let us show that the Tamari lattice with 14 vertices has a wild representation type by exhibiting a wild subquiver.



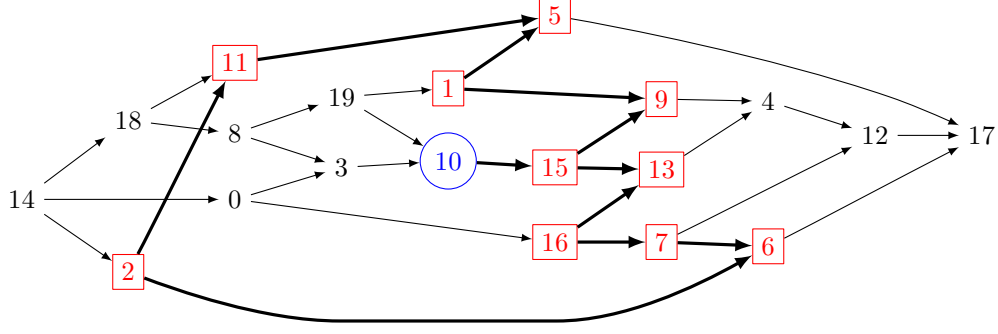
The induced quiver-with-relations on the marked vertices is just a quiver, as one can check that no relation is implied by the commuting relations in the initial Hasse diagram. Removing the vertex 2 in this quiver gives a quiver of affine type $\mathbb{A}_7^{(1)}$.

Here is the similar wild sub-quiver for the other cambrian lattice of type \mathbb{A}_3 .



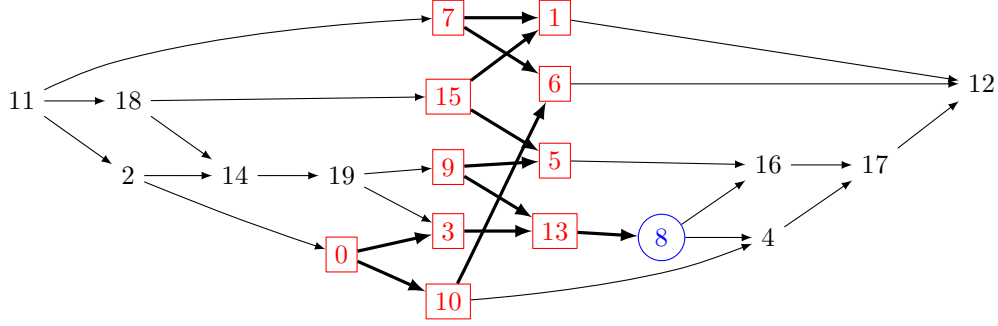
3.4 Type \mathbb{B}_3

Let us now consider one of the cambrian lattices of type \mathbb{B}_3 , with 20 vertices. Let us again show that this has wild representation type by exhibiting a wild subquiver.



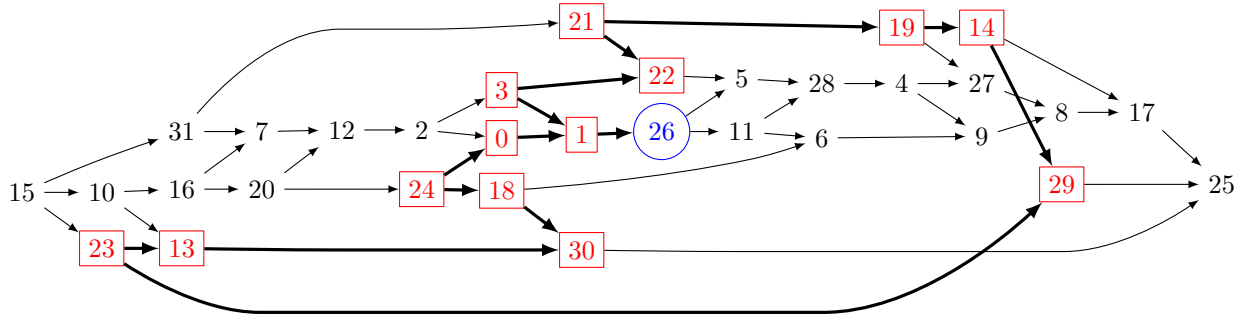
The induced quiver-with-relations on the marked vertices is just a quiver, as one can check that no relation is implied by the relations in the initial Hasse diagram. Removing the vertex 10 in this quiver gives a quiver of affine type $\mathbb{A}_9^{(1)}$.

Here is the similar wild sub-quiver for the other cambrian lattice of type \mathbb{B}_3 .



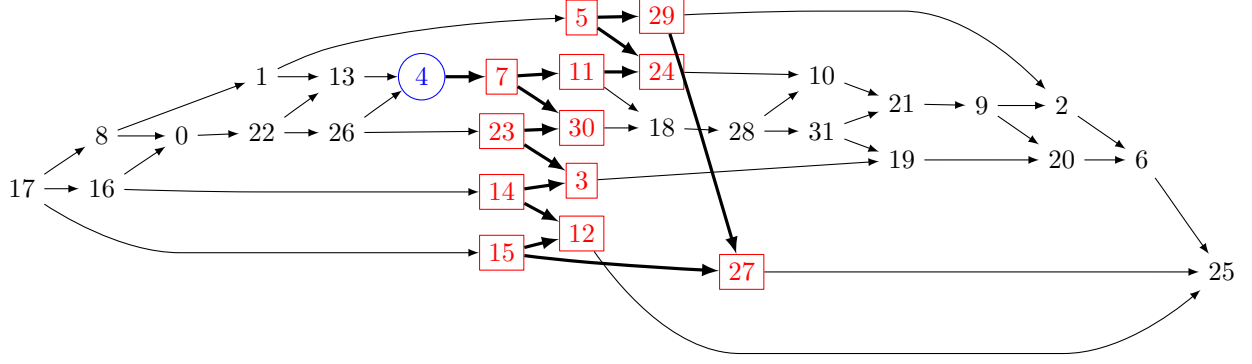
3.5 Type \mathbb{H}_3

Let us now consider one of the cambrian lattices of type \mathbb{H}_3 , with 32 vertices. Let us again show that this has wild representation type by exhibiting a wild subquiver.



The induced quiver-with-relations on the marked vertices is just a quiver, as one can check that no relation is implied by the relations in the initial Hasse diagram. Removing the vertex 26 in this quiver gives a quiver of affine type $\mathbb{A}_{12}^{(1)}$.

Here is the similar wild sub-quiver for the other cambrian lattice of type \mathbb{H}_3 .



Remark 3.7. Note the similarity in the shape of the wild subquivers for \mathbb{A}_3 , \mathbb{B}_3 and \mathbb{H}_3 . They all consist of something like a middle belt between a left part and a right part, plus one extra vertex. They have been obtained by carefully removing well-chosen maximal or minimal elements, until reaching this kind of configuration.

3.6 Weak order on the Coxeter group

The cambrian lattice $\text{Cambrian}(W, c)$ is known to be a quotient of the weak order on W such that the fibers of the quotient are intervals in the weak order. By Proposition 2.2, when the rank of W is at least 3 and W is not $\mathbb{A}_1 \times \mathbb{A}_1 \times \mathbb{A}_1$, the weak order poset on W has a wild representation type. When $W = \mathbb{A}_1 \times \mathbb{A}_1 \times \mathbb{A}_1$, then the weak order poset is isomorphic to the corresponding cambrian lattice, which is a cube. In particular, it has a tame representation type. Finally, it is easy to check that when the rank of W is at most 2, then the weak order poset has a finite representation type. Indeed, in these cases, the posets are obtained by adding a minimal and a maximal element to the disjoint union of two chains. These posets are known to be of finite representation type (see for instance [Cha69]).

4 Antichain posets

This is another family of posets, expected but not known to be derived equivalent to cambrian lattices.

Let Φ be a finite root system. The set Φ_+ of positive roots in Φ is endowed with a partial order by the relation $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a positive linear combination of simple positive roots. This poset is called the root poset of Φ . If n is the rank of the root system, this poset has n minimal elements, the simple positive roots.

Given a finite root system Φ , the poset $\text{NonNesting}(\Phi)$ is defined as the lattice of order ideals in the root poset of Φ . Recall that an order ideal in a poset P is a subset L of P such that $x \in L$ and $y \leq x$ implies $y \in L$. The partial order on the set of order ideals is given by inclusion.

For S a subset of Φ_+ , let us denote by $L(S)$ the order ideal generated by S . Every order ideal can be uniquely written as $L(S)$ where S is an antichain in the root poset.

The poset $\text{NonNesting}(\Phi)$ has a unique minimal element, the empty order ideal. It also has a unique maximal element, the full root poset. Moreover the restricted poset on the set of order ideals contained in $L(\alpha_1, \dots, \alpha_n)$ is isomorphic to the n -cube poset.

4.1 Easy types

If the rank $n \leq 2$, then every poset $\text{NonNesting}(\Phi)$ has a finite representation type. Indeed, in this case Hasse diagram of the poset is a commutative square with a (possibly empty) chain attached to the maximal element of the square. By Proposition 2.4, it is derived equivalent to a Dynkin diagram of type \mathbb{D} . The result follows from Proposition 2.3.

If the rank $n \geq 4$, then $\text{NonNesting}(\Phi)$ contains a 4-cube, hence one can apply Lemma 3.3 to get that representation type is wild.

There remains only to handle the cases of rank 3.

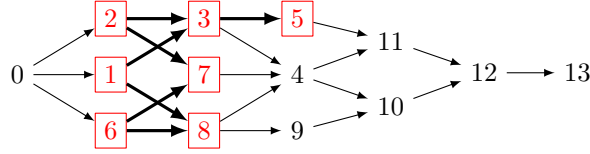
4.2 Types of rank 3

For the reducible type $(\mathbb{A}_1)^3$, the poset $\text{NonNesting}(\Phi)$ is isomorphic to the cube poset. By Lemma 3.5, it has a tame representation type.

Assume now that Φ is not of type $(\mathbb{A}_1)^3$. Then there exist at least two simple roots (say α_1 and α_2) in the root poset that are covered by a common root β . Then β covers only α_1 and α_2 . This property also holds for the substitute of the root poset in type \mathbb{H}_3 introduced by D. Armstrong in [Arm09, §5.4.1].

The Hasse diagram of the poset $\text{NonNesting}(\Phi)$ contains a cube, as its restriction to the order ideals contained in $L(\alpha_1, \alpha_2, \alpha_3)$. The order ideal $L(\beta)$ covers $L(\alpha_1, \alpha_2)$ and no other vertex of the cube.

Consider the induced poset Q on the vertices $L(\alpha_i)$, $L(\alpha_i, \alpha_j)$ and $L(\beta)$. This is just a quiver, with no commuting relation. Removing $L(\beta)$ gives an affine quiver of type $\mathbb{A}_5^{(1)}$. Therefore Q has wild representation type, and so does $\text{NonNesting}(\Phi)$.



5 Stokes lattices

Let us turn to another family of posets, containing the cambrian lattices of type \mathbb{A} . The Stokes lattices were introduced by the first author as posets in [Cha16] (inspired by previous work of Baryshnikov in [Bar01]) and they were proved to be lattices by Garver and McConville in [GM16].

Since the article [GM16], the Stokes lattices can even be considered as a special case of a more general class of lattices, attached to dissections of polygons. Our arguments below work just the same for this extended class, which has very similar properties.

Just as the cambrian lattices, the Stokes lattices also have n -regular Hasse diagrams, where n is a parameter called the rank of the Stokes lattice.

5.1 Easy types

There are essentially the same easy cases as in the Cambrian setting.

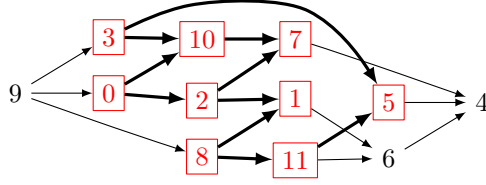
In rank $n \leq 2$, the only possible Stokes posets are cambrian lattices of type \mathbb{A} , already considered before.

In rank $n \geq 4$, one can also use Lemma 3.3 to get wild representation type.

So once again, there remains to handle the case of rank 3.

5.2 Stokes lattices of rank 3

Excluding the intersection with the cambrian cases of type \mathbb{A} (reducible or not), there remains only one case to consider, which is a poset with 12 vertices and the following shape.



By Proposition 2.5, the subquiver on vertices $\{0, 1, 2, 3, 5, 7, 8, 10, 11\}$ and its commutative square $(0, 2, 10, 7)$ is of wild representation type. So, the poset is of wild representation type.

References

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